ON A MAGNETO-ELASTIC SYSTEM WITH DISCONTINUOUS COEFFICIENTS AND THE PROPAGATION OF A WEAK DISCONTINUITY*

Andrea Donato**

SOMMARIO. In questa Nota si deduce un sistema di equazioni per la Magneto-elasticità con deformazioni finite nel caso tridimensionale. Come Applicazione si studia la propagazione di una discontinuità debole nella ipotesi che i coefficienti delle equazioni risultano essi stessi discontinui. Tale ipotesi risulta verificata, ad es., quando si considerano due differenti mezzi magneto-elastici uno in contatto con l'altro.

SUMMARY. In this paper we deduce the magneto-elastic system of equations in the three-dimensional case. As an application we study the propagation of a weak discontinuity when a strong discontinuity also occurs.

INTRODUCTION.

Many papers concerned with the description of magneto-elastic phenomena are founded on a system of partial differential equations that is obtained by coupling the Maxwell equations with Hooke's law (see [1] also for references). This system of equations becomes hyperbolic when an infinite conductivity is considered, so it then becomes appropriate to study wave propagation [2] (for other detailed references see [3]). However, in all these papers, the nonlinearity of the system is caused solely by the electromagnetic field, because of the linearity of the Hooke's law that is used to describe the small amplitude deformations.

Recently, a perfect magneto-elastic medium subject to a finite deformation in the one-dimensional case has been considered. Essentially, this work examined the propagation of weak discontinuities, shocks and simple wave motions [3], [4]. The system of equations used in these papers was obtained in [4] by direct consideration of the case of a one-dimensional deformation using a method which cannot be generalised to the three-dimensional case. The present paper has two distinct aims: the first one is to obtain a system of equations describing the motion of an electrically conducting elastic medium subject to a finite three-dimensional deformation in a varying magnetic field: the second one is to extend the analysis made in [3] and [4] by considering the propagation of a weak discontinuity when the system of equations has discontinuous coefficients [5], [12]. This situation may occur, for instance, when there are two different elastic media in contact with one another.

So, in the first part of the paper we obtain, following the usual procedure of continuum mechanics, the required system of equations in various forms, as well as the case of infinite conductivity. This last case gives rise to a hyperbolic quasilinear system of equation in conservative form. If we particularise in the case of one-dimensional deformation we obtain exactly the system considered in [3], [4].

In the last part of the paper we are able to study the effect of a strong discontinuity across a line $\mathscr{D}$ on a transmitted and a reflected weak discontinuity that remains completely determined.

1. BASIC EQUATIONS IN LAGRANGIAN FORM.

We consider an electrically conducting elastic medium, subject to finite deformation, in a varying magnetic field. The electromagnetic field influences the elastic field by entering the elastic stress equations of motion as a body force called the Lorentz force and the elastic field influences the electromagnetic field by modifying Ohm's law. If C^* and C represent, respectively, the natural configuration of the body (without stress) and the actual configuration, then we denote by u_i the components of the displacement vector, so that

$$x_i = y_i + u_i \qquad (i = 1, 2, 3), \tag{1}$$

where y_i are the co-ordinates of an arbitrary point P^* of C^* and x_i are the co-ordinates of an arbitrary point P of C with respect a rectangular cartesian frame T. We assume [1], [2] that between the two possible configurations C^* and C there is a one-to-one mapping which sets P into correspondence with P^* so that

$$x_i = x_i(y_1, y_2, y_3, t). \tag{2}$$

The relation (2) satisfies for every measurable set in C^*, including the boundary, the usual conditions [6]; in particular we have:

$$y_k = y_k(x_i, t) \tag{3}$$

$$D = \det\left|\frac{\partial x_i}{\partial y_k}\right| > 0.$$

We define the deformation gradients as

$$x_{i,k} = \frac{\partial x_i}{\partial y_k} = \delta_{ik} + u_{i,k}, \tag{4}$$

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** Istituto matematico dell'Università di Messina.

$$y_{i/k} = \frac{\partial y_i}{\partial x_k} = \delta_{ik} - u_{i,k}, \tag{5}$$

and the strain tensor through the relation

$$\epsilon_{ik} = \frac{1}{2}(x_{r,i}\, x_{r,k} - \delta_{ik}) = \frac{1}{2}(u_{i,k} + u_{k,i} + u_{r,i}\, u_{r,k}), \tag{6}$$

and the Green's deformation tensor by

$$G_{ik} = x_{r,i}\, x_{r,k} = \delta_{ik} + 2\epsilon_{ik}. \tag{7}$$

Using the usual procedure of continuum mechanics is easy to see that the Eulerian expression for the fundamental equations governing the motion are:

$$\rho a_i = \phi_{ik/k} + E_{ilm} J_l B_m + \rho_{(e)} E_i + \rho f_i, \tag{8}$$

where ρ is the mass density in the deformed material, a_i is the acceleration of P, ϕ_{ik} the Cauchy stress tensor, J and B are, respectively, the electric current density and the magnetic induction, so that the second term on the right-hand side of (1) is the Lorentz force, E_{ilm} is a Ricci tensor, $\rho_{(e)}$ is the electric charge density, E is the electric field vector and f is the body force per unit mass. In addition to equation (8) we must consider the Maxwell equations governing the electromagnetic field. In order to give a Lagrangian description of the magneto-elastic system it is more convenient to write the Maxwell equations in integral form. The Faraday's law for a conducting medium moving with velocity v can be written as [3]

$$-\frac{d}{dt}\int_{\sigma} B_k\, d\sigma^k = \int_{\gamma} (E_k + E_{klm} v_l B_m)\, dx^k, \tag{9}$$

and Ampere's law, when we neglect the desplacement current, is given by the relation

$$\int_{\sigma} J_k\, d\sigma^k = \int_{\gamma} H_k\, dx^k. \tag{10}$$

Moreover we consider the following constitutive relations

$$\mathbf{D} = \epsilon \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H}, \tag{11}$$

and the Ohm's law for a moving conductor

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{12}$$

where ϵ, μ and σ represent, respectively, the electric permittivity, the magnetic permeability and the electrical conductivity. If we consider an infinitely conducting non-magnetizable elastic solid and we neglect the electric charge density, the desplacement current and the body force, the basic system of equation for magneto-elasticity in conservative Eulerian form becomes [3], [4].

$$\rho a_i = \phi_{ik/k} + M_{ik/k},$$

$$\frac{\partial B_i}{\partial t} = (v_i B_k - v_k B_i)_{/k},$$

$$B_{i/i} = 0,$$

$$M_{ik} = \frac{1}{\mu}\left(B_i B_k - \frac{1}{2} B_l B_l \delta_{ik}\right).$$

In order to deduce the corresponding Lagrangian form, which is more convenient for our future work, we observe that equations (9) and (10) can be written in the form

$$-\frac{d}{dt}\int_{\bar\sigma} B_k C_{kl}\, d\bar\sigma^l = \int_{\gamma} (E_k + E_{ksm} v_s B_m) x_{k,l}\, dy^l \tag{9}$$

$$\int_{\bar\sigma} J_k C_{kl}\, d\bar\sigma^l = \int_{\bar\gamma} H_k x_{k,l}\, dy^l, \tag{10}$$

where we have used the following relation between the deformed and the undeformed medium

$$n^k\, d\sigma = d\sigma^k = C_{kl}\, d\bar\sigma^l \qquad C_{kl} = \text{cofactor of } x_{k,l} = D y_{l/k}$$
$$dx_k = x_{k,l}\, dy_l. \tag{14}$$

If we put

$$\bar B_l = C_{kl} B_k,$$
$$\bar E_l = (E_k + E_{ksm} v_s B_m) x_{k,l},$$
$$\bar J_l = C_{kl} J_l,$$
$$\bar H_l = H_k x_{k,l},$$

the localization of eqs. (9') and (10') gives us

$$-\frac{\partial \bar B_l}{\partial t} = E_{lpq} \bar E_{q,p}, \tag{16}$$

$$\bar J_l = E_{lpq} \bar H_{q,p}, \tag{17}$$

which we can take as the Lagrangian form of Maxwell's equations. The quantities $\bar B_l$, $\bar J_l$, $\bar E_l$ and $\bar H_l$ may then be regarded as the «Lagrangian» field quantities. The constitutive relations (11) and Ohm's law, in terms of Lagrangian quantities, now become

$$\bar B_k = \mu D \bar H_l G^{-1}_{lk} \qquad \text{with} \qquad G^{-1}_{lk} G_{kr} = \delta_{lr} \tag{18}$$

$$\bar D_k = \epsilon D \bar E_l G^{-1}_{lk} \tag{19}$$

$$\bar J_k = \sigma D \bar E_l G^{-1}_{lk}. \tag{20}$$

To obtain the Lagrangian form of eq. (8) we first introduce the Piola-Kirchoff stress tensor T_{kl} that is related to the Cauchy stress tensor ϕ_{ik} by the relation

$$T_{kl} = D\phi_{il}\, y_{k/i}, \tag{21}$$

while the Lagrangian Cauchy stress tensor is given by

$$Y_{lm} = D\phi_{ik}\, y_{l/i}\, y_{m/k} = T_{ls}\, y_{m/s}. \tag{22}$$

Using the standard procedure [7] for deducing the Lagrangian expression for the continuum mechanics equations we find the following expression for the material form of eq. (8):

$$\bar\rho \ddot u_i = T_{kl,k} + E_{ilm} D J_l B_m + \bar\rho_{(e)} E_i + \bar\rho f_i, \tag{23}$$

with

$$\bar\rho = \rho D, \qquad \bar\rho_e = \rho_{(e)} D. \tag{24}$$

When we neglect the displacement current we know that $\mathbf{J} = \operatorname{curl} \mathbf{H}$, so

$$DE_{ilm} J_l B_m = DM_{ik,l}\, y_{l/k} = (C_{kl} M_{ik})_{,l}, \tag{25}$$

and we can take for the material form of Maxwell's tensor the following

$$\bar{M}_{il} = M_{ik} C_{kl}. \tag{26}$$

From (16), (17), (23), (25) and (26) we find the following conservative system, in Lagrangian form, governing the magneto-elastic interaction when $\sigma \to \infty$, $\rho_{(e)} = 0$, $f_i = 0$ and the displacement current is neglected,

$$\begin{cases} \bar{\rho}\ddot{u}_i = T_{ki,k} + \bar{M}_{ik,k} \\ \dfrac{\partial \bar{B}_l}{\partial t} = 0, \qquad \bar{B}_{l,l} = 0 \\ \bar{M}_{ik} = \left(\bar{B}_k \bar{H}_s - \dfrac{1}{2}\bar{B}_l \bar{H}_l \delta_{ks}\right) y_{s/i}. \end{cases} \tag{27}$$

To this system must be added the constitutive relation (18).

The same system written with Eulerian field quantities, but expressed in Lagrangian co-ordinates, becomes:

$$\begin{cases} \bar{\rho}\ddot{u}_i = T_{ki,k} + \bar{M}_{ik,k} \\ \dfrac{\partial (DB_i)}{\partial t} - B_k C_{kl} \dfrac{\partial v_i}{\partial y_l} = 0, \qquad (C_{kl} B_k)_{,l} = 0 \\ \bar{M}_{il} = \dfrac{1}{\mu}\left(B_i B_k - \dfrac{1}{2} B_l B_l \delta_{ik}\right) C_{kl}, \qquad \mathbf{B} = \mu \mathbf{H}. \end{cases} \tag{28}$$

We observe that by integration of eq. $(27)_2$ from some initial time t_0 to $t > t_0$ we have

$$\bar{B}_l = \bar{B}_{0l}(y_1, y_2, y_3, t_0). \tag{29}$$

At this point, it is possible to consider, using eq. (29) to eliminate the variable $\bar{B}_l$ and, because of (18), also the variable $\bar{H}_l$ from the system. However, this procedure is not convenient [6] because equation (29) is not immediately applicable to domains in which discontinuities are present. Moreover, the use of eq. (29) introduces the initial conditions into the system and complicates the application of the general theory of conservation laws.

2. PROPAGATION OF A WEAK DISCONTINUITY IN THE ONE-DIMENSIONAL CASE WHEN THE SYSTEM HAS DISCONTINUOUS COEFFICIENTS.

2.1. Statement of problem, the propagation velocities of weak discontinuities, left and right eigenvectors.

If we make the assumption that all the field quantities depend on $y_1 = y$ and t we have

$$x_i = y_i + u_i(y, t) \tag{30}$$

$$D = 1 + u'_1 \tag{31}$$

$$C_{kr} = \begin{vmatrix} 1 & -u'_2 & -u'_3 \\ 0 & 1 + u'_1 & 0 \\ 0 & 0 & 1 + u'_1 \end{vmatrix}, \tag{32}$$

and the system (28) assumes the following form

$$\begin{aligned} & B_1 = \text{const. independent of } y \text{ and } t, \\ & \partial_t\{(1 + w_1)B_i\} - B_1 v'_i, \\ & \bar{\rho}\partial_t v_i = T'_{1i} + M'_{1i}, \\ & \partial_t w_k = v'_k, \\ & M'_{1i} = \frac{1}{\mu}\{B_1 B'_i - (B_2 B'_2 + B_3 B'_3)\delta_{1i}\}, \end{aligned} \tag{33}$$

which coincides exactly with the system used by J. Bazer & W. B. Ericson in [3].

If we assume, moreover, a perfect hyperelastic, isotropic material with reversible and isothermal transformation, without electrostrictive and magnetostrictive effect, we can suppose that the following constitutive relations are valid [3].

$$T_{1i} = \bar{\rho}\, \frac{\partial \mathscr{F}(D, N)}{\partial w_i}, \tag{34}$$

where

$$N = w_2^2 + w_3^2, \qquad D = 1 + w_1. \tag{35}$$

In what follows we regard $\bar{\rho}$ as everywhere constant.

The system (33) with (34), is a nonlinear system of eight equations for eight quantities $(B_2, B_3, D, w_2, w_3, v_1, v_2, v_3)$ and in matrix notation can be written

$$\partial_t \mathbf{U} + A\partial_y \mathbf{U} = 0, \tag{36}$$

where

$$\mathbf{U} = \begin{bmatrix} B_2 \\ B_3 \\ D \\ w_2 \\ w_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}, \qquad A = \left[\begin{array}{c|c} 0 & A_1 \\ \hline A_2 & 0 \end{array}\right],$$

$$A_1 = \begin{bmatrix} \dfrac{B_2}{D} & -\dfrac{B_1}{D} & 0 \\ \dfrac{B_3}{D} & 0 & -\dfrac{B_1}{D} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{37}$$

$$A_2 = \begin{bmatrix} \dfrac{B_2}{\bar{\rho}\mu} & \dfrac{B_3}{\bar{\rho}\mu} & -\mathscr{F}_{DD} & -2\mathscr{F}_{DN} w_2 & -2\mathscr{F}_{DN} w_3 \\ -\dfrac{B_1}{\bar{\rho}\mu} & 0 & -2\mathscr{F}_{ND} w_2 & -4\mathscr{F}_{NN} w_3^2 & -4\mathscr{F}_{NN} w_2 w_3 \\ 0 & -\dfrac{B_1}{\bar{\rho}\mu} & -2\mathscr{F}_{ND} w_3 & -4\mathscr{F}_{NN} w_2 w_3 & -4\mathscr{F}_{NN} w_3^2 + 2\mathscr{F}_N \end{bmatrix}$$

or, in conservative form, as

$$\partial_t \mathbf{V} + \partial_y \mathbf{F} = 0, \tag{38}$$

with

$$\mathbf{V} = \begin{bmatrix} DB_2 \\ DB_3 \\ D \\ w_2 \\ w_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} -B_1 v_2 \\ -B_1 v_3 \\ -v_1 \\ -v_2 \\ -v_3 \\ -\left\{\mathcal{F}_D + \frac{1}{\bar{\rho}\mu}\left(B_1^2 - \frac{1}{2}B^2\right)\right\} \\ -\left(2\mathcal{F}_N w_2 + \frac{1}{\bar{\rho}\mu} B_1 B_2\right) \\ -\left(2\mathcal{F}_N w_3 + \frac{1}{\bar{\rho}\mu} B_1 B_3\right) \end{bmatrix}. \tag{39}$$

Now we suppose that the coefficients of the describing eqs. (36) and (39) are piecewise continuous functions, with their discontinuities occurring across a straight line $\mathcal{D}$ of (y, t) plane [5]. To be precise, we consider $\mathcal{D}$ at rest and orthogonal to the y axis so that in the (y, t) plane it has the equation $y = y_d$. Moreover, we denote by $\mathcal{R}_u$ the determinancy domain of $\mathbf{U}$ corresponding to the dependence domain

$$\mathcal{T} = \{y \mid 0 < y < y_d, t = 0\}$$

on the left of y_d. While on the right of y_d we suppose that the governing system is

$$\partial_t \mathbf{U}^* + \mathbf{A}^* \partial_y \mathbf{U}^* = 0 \tag{40}$$

or

$$\partial_t \mathbf{V}^* + \partial_y \mathbf{F}^* = 0, \tag{41}$$

where A^*, $\mathbf{U}^*$, $\mathbf{V}^*$ and $\mathbf{F}^*$ are similar to the corresponding quantities defined in (37) and (39). In the same way we denote by $\mathcal{R}_{u^*}$ the determinancy domain of $\mathbf{U}^*$ corresponding to the dependence domain $\mathcal{T}^* = \{y \mid y_d < y, t = 0\}$.

In this situation the propagation characteristics of weak discontinuity (i.e. continuous vector function with bounded discontinuities in its first order partial derivatives across the wavefront trace) moving along the wavefront trace became modified after crossing the line $\mathcal{D}$ and the general theory has been extensively studied by A. Jeffrey [5], [10], [12].

In order to study an application of this theory to the magnetoelastic system (33) we consider, for simplicity, the special case in which $w_3 = 0$ and $B_3 = 0$ [3]. In this case the eigenvalues of matrix A are

$$\begin{aligned} &\lambda = 0 \text{ with multiplicity } m = 2, \\ &\lambda_S^{(\pm)} = \pm\sqrt{(a - \sqrt{\Delta})}, \\ &\lambda_I^{(\pm)} = \pm\sqrt{2\mathcal{F}_N + \frac{B_1^2}{\bar{\rho}\mu D}}, \\ &\lambda_F^{(\pm)} = \pm\sqrt{(a + \sqrt{\Delta})}, \end{aligned} \tag{42}$$

in which

$$\Delta = \left(\mathcal{F}_{DD} + \frac{B_2^2}{\bar{\rho}\mu D} - 2\mathcal{F}_N - 4\mathcal{F}_{NN} - \frac{B_1^2}{\bar{\rho}\mu D}\right) + + 8\mathcal{F}_{ND} w_2 - 4\frac{B_1 B_2}{\bar{\rho}\mu D} > 0 \tag{43}$$

$$a = \mathcal{F}_{DD} + 2\mathcal{F}_N + 4\mathcal{F}_{NN} w_2^2 + \frac{B_1^2 + B_2^2}{\bar{\rho}\mu D},$$

and $\lambda^{(+)}$ corresponds to a wave travelling to the right and $\lambda^{(-)}$ to one travelling to the left. It is easy to see [3], [9] that the following inequalities are valid

$$\lambda_S^{(+)} < \lambda_I^{(+)} < \lambda_F^{(+)}.$$

The corresponding left and right eigenvectors are

(i) *for the case* $\lambda = 0$

$$\begin{cases} \mathbf{l}_0^{(4,1)} = \left(0, 1, 0, 0, -\frac{B_1}{D}, 0, 0, 0\right), \\ \mathbf{l}_0^{(4,2)} = \left(1, 0, \frac{B_2}{D}, -\frac{B_1}{D}, 0, 0, 0, 0\right), \end{cases} \tag{44}$$

$$\begin{cases} \tilde{\mathbf{r}}_0^{(4,1)} = \left(0, B, 0, 0, \frac{B_1 B_2}{2\mathcal{F}_N}, 0, 0, 0\right), \\ \tilde{\mathbf{r}}_0^{(4,2)} = \left(B, 0, -\frac{BT}{M}, \frac{BP}{M}, 0, 0, 0, 0\right), \end{cases} \tag{45}$$

($\tilde{}$ denotes the transposition operator).

(ii) *for the case* $\lambda = \lambda_S^{(+)}$ *and* $\lambda = \lambda_F^{(+)}$, *respectively*,

$$\begin{cases} \mathbf{l}_0^{(S)} = \left(\frac{DK_S}{\bar{\rho}\mu S}, \frac{B_2 K_S}{\bar{\rho}\mu S}, -\lambda_S^{(\pm)}, \frac{B_1 K_S}{\bar{\rho}\mu S}, \frac{\lambda_S^{(\pm)} K_{1S}}{S}, 0, 1, -\frac{K_{1S}}{S}\right), \\ \mathbf{l}_0^{(F)} = \left(\frac{DK_F}{\bar{\rho}\mu S}, \frac{B_2 K_F}{\bar{\rho}\mu S}, -\lambda_F^{(\pm)}, -\frac{B_1 K_F}{\bar{\rho}\mu S}, \frac{\lambda_F^{(\pm)} K_{1F}}{S}, 0, 1, -\frac{K_{1F}}{S}\right), \end{cases} \tag{46}$$

$$\begin{cases} \tilde{\mathbf{r}}_0^{(S)} = \left(-\frac{K_S}{S}, 0, 1, -\frac{K_{1S}}{S}, 0, -\lambda_S^{(\pm)}, \frac{\lambda_S^{(\pm)} K_{1S}}{S}, 0\right), \\ \tilde{\mathbf{r}}_0^{(F)} = \left(-\frac{K_F}{S}, 0, 1, -\frac{K_{1F}}{S}, 0, -\lambda_F^{(\pm)}, \frac{\lambda_F^{(\pm)} K_{1F}}{S}, 0\right). \end{cases} \tag{47}$$

(iii) *for the case* $\lambda = \lambda_I$

$$\mathbf{l}_0^{(I)} = \left(0, 1, 0, 0, \frac{2\mathcal{F}_N \bar{\rho}\mu}{B_1}, 0, 0, -\frac{\lambda_I^{(\pm)} \bar{\rho}\mu}{B_1}\right), \tag{48}$$

$$\tilde{r}_0^{(1)} = \left(0, B, 0, 0, \frac{BD}{B_2}, 0, 0, -\frac{\lambda_I^{(\pm)} BD}{B_1}\right), \tag{49}$$

where we have set

$$\begin{cases} T = B_2(2\mathscr{F}_N + 4\mathscr{F}_{NN} w_2^2) + 2B_1 \mathscr{F}_{ND} w_2, \\ P = B_1 \mathscr{F}_{DD} + 2B_2 \mathscr{F}_{ND} w_2, \\ M = \mathscr{F}_{DD}(2\mathscr{F}_N + 4\mathscr{F}_{NN} w_2^2) - 4\mathscr{F}_{ND} w_2^2, \\ K = \dfrac{(\lambda^2 - \mathscr{F}_{DD})B_1 - 2B_2 \mathscr{F}_{ND} w_2}{D}, \\ K_1 = \lambda^2 - \mathscr{F}_{DD} - \dfrac{B_2^2}{\bar{\rho}\mu D}, \\ S = \dfrac{B_1 B_2}{\bar{\rho}\mu D} - 2\mathscr{F}_{ND} w_2. \end{cases} \tag{50}$$

From (42) and (49) it follows immediately that the wave with velocity of propagation $\lambda = \lambda_I$ is exceptional in the sense of Lax, while the other waves are not, in general, exceptional.

We see that there are 7 distinct families of characteristic curves in the determinacy domain $\mathscr{R}_u$ determined as the solutions to the equations

$$\frac{dy}{dt} = \lambda^{(i)}, \qquad i = 1, 2, \ldots, 7, \tag{51}$$

$\lambda^{(1)} = \lambda_F^{(-)}$, $\lambda^{(2)} = \lambda_I^{(-)}$, $\lambda^{(3)} = \lambda_S^{(-)}$, $\lambda^{(4)} = 0$, $\lambda^{(5)} = \lambda_F^{(4)}$, $\lambda^{(6)} = \lambda_I^{(+)}$, $\lambda^{(7)} = \lambda_S^{(+)}$ and subject to appropriate initial conditions. In the case of matrix A^* there are also 7 distinct families of characteristic curves in the determinacy domain $\mathscr{R}_{u^*}$ determined as solutions to the equations.

$$\frac{dy}{dt} = \lambda^{*(i)}, \qquad i = 1, 2, \ldots, 7, \tag{52}$$

subject to appropriate initial conditions. The eigenvalues $\lambda^{*(i)}$ of A^* and the corresponding left and right eigenvectors are identical to those above when we add the symbol * to the quantities that define them. We assume that the propagating weak discontinuity starts at a point $(y_0, 0) \in \mathscr{T}$ and denote by $\varphi(y, t) = 0$ the equation of the characteristic in $\mathscr{R}_u$ through $(y_0, 0)$ forming the initial wavefront trace. We denote by $\varphi^*(y, t) = 0$ the equation of the new wavefront trace in $\mathscr{R}_{u^*}$ through P, the point where $\varphi(y, t) = 0$ meets $\mathscr{D}$.

Moreover, we state the initial value problem in terms of the fastest weak discontinuity in U propagating initially to the right in $\mathscr{R}_u$ and consequently moving along the characteristic $C^{(5)}$ with equation $\varphi(y, t) = 0$ which passes through the point $(y_0, 0)$, so that $\lambda^{(5)} = \lambda_F$ corresponds to the wavefront trace. Similarly, we assume that $\varphi^*(y, t) = 0$ is the equation of the fastest weak discontinuity in U^* along the characteristic $C^{*(\varphi^*)} = C^{*(5)}$ determined by

$$\frac{dy}{dt} = \lambda^{*(5)},$$

with the requirement that $C^{*(\varphi^*)}$ passes through the point $P \equiv (y_p, t_p)$.

In the next section, using the general theory formulated in [5] and [11], we determine the propagation characteristics of the weak discontinuity in U along C^* when subjected to the influence of the strong discontinuity at P on $\mathscr{D}$.

2.2. Propagation of a weak discontinuity when a strong discontinuity occurs in the coefficients of the system.

Our system is homogeneous, and the coefficient matrices do not explicitly depend on y and t, so that some simplification is possible. With the usual procedure ([5], [9], [10]), denoting by Π the jump of $U = \partial U/\partial\varphi$ across $C^{(\varphi)}$ and by Π^* the jump of $U^*_{\varphi^*}$ across $C^{*(\varphi)}$ we have [9]

$$\begin{aligned} \Pi &= \tilde{\Pi} = \tilde{U}_\varphi = \text{const.}, \\ \Pi^* &= \tilde{\Pi}^* = \tilde{U}^*_{\varphi^*} = \text{const.} \end{aligned} \tag{53}$$

The each equal their initial value, respectively, in $\mathscr{R}_u$ and $\mathscr{R}_{u^*}$. Hence we conclude that [5],

$$U_y = \tilde{U}_\varphi \varphi_y, \qquad U^*_y = U^*_{\varphi^*} \varphi_{y^*}. \tag{54}$$

Moreover, because A is independent of y and t, the solution vector $U(y, t) = U_0$ to equation (36) ahead of the wavefront trace $C^{(\varphi)}$ is constant, and similarly for $U^*(y, t) = U^*_0$, the solution of equation (40) in $\mathscr{R}_{u^*}$ ahead of the wavefront trace $C^{*(\varphi^*)}$. In this case we know that Π is constant and equal to its initial value, and that is satisfies the algebraic equation

$$l_0^{(j)} \Pi = 0, \qquad (j \neq 5). \tag{55}$$

This implies that Π is proportional to the right eigenvector corresponding to $\lambda = \lambda^{(5)} = \lambda_F^{(+)}$. Since the system (36) and (40) may be expressed in the conservation form (38) and (41), it follows that when $\tilde{\lambda} = 0$ with $\tilde{\lambda}$ the speed of propagation of $\mathscr{D}$, we obtain

$$\{F\}_{\varphi^*} - \{F\}_{\varphi^*} = 0, \qquad 0\nu \tag{56}$$

$$\begin{cases} B_1 = B_1^*, \\ \nu_1 = \nu_1^*, \\ \nu_2 = \nu_2^*, \\ \nu_3 = \nu_3^*, \\ \mathscr{F}_D + \dfrac{1}{\bar{\rho}\mu}\left(B_1^2 - \dfrac{1}{2}B^2\right) = \\ = \mathscr{F}^*_{D^*} + \dfrac{1}{\bar{\rho}^*\mu^*}\left(B_1^{*2} - \dfrac{1}{2}B^{*2}\right), \\ 2\mathscr{F}_N w_2 + \dfrac{1}{\bar{\rho}\mu} B_1 B_2 = 2\mathscr{F}^*_{N^*} w_2^* + \dfrac{1}{\bar{\rho}^*\mu^*} B_1^* B_2^*. \end{cases} \tag{56'}$$

In order to obtain the relationship between the gradients of U and U* across $\mathscr{D}$ we first observe that immediately to the left of $\mathscr{D}$ at P there are 5 characteristics belonging to system (36). Only $\lambda_F^{(-)} = \lambda^{(1)}$, $\lambda_I^{(-)} = \lambda^{(2)}$, $\lambda_S^{(-)} = \lambda^{(3)}$, as time increases, radiate out from P and enter $\mathscr{R}_u$ to the left of $\mathscr{D}$, whilst the two characteristics corresponding to $\lambda = 0$ are tangent to the line $\mathscr{D}$. The remaining three characteristic belonging to system (36) will lie to the right or $\mathscr{D}$ and, if produced, would enter $\mathscr{R}_{u*}$. Analogously, immediately to the right of $\mathscr{D}$ at P there will be 5 characteristics belonging to system (40) which, as time increases, radiate out to the right from P and enter $\mathscr{R}_{u*}$. These correspond to $\lambda_F^{*(+)} = \lambda^{*(5)}$, $\lambda_I^{*(+)} = \lambda_S^{*(6)}$, $\lambda^{*(+)} = \lambda^{*(7)}$ whilst the two characteristics corresponding to $\lambda^* = 0$ are tangent to $\mathscr{D}$. We denote by $U_\varphi^{(R)}$ the value of the vector U_φ in $\mathscr{R}_u$ immediately to the right of the characteristic curve corresponding to $\lambda = 0$, and by $U_{\varphi^*}^{*(T)}$ the transmitted vector $U_{\varphi^*}^*$ in $\mathscr{R}_{u*}$ to the left of the characteristic $\lambda^* = 0$ and ahead of $\mathscr{D}$ (fig. 1).

This allows us to write the relations (in this case $U_{0y} = U_{0y}^* = 0$) at P [11]:

$$U_y^{(R)} = \Pi\varphi_y + \sum_{i=1}^{3} \Pi_i^{(R)}\varphi_{iy}; \qquad U_y^{*(T)} = \sum_{i=1}^{3} \Pi_i^{*(T)}\varphi_{iy}^*. \tag{57}$$

These are connected across $\mathscr{D}$ by the equation [5], [11]:

$$\{(\nabla_u \mathbf{F})(A U_y^{(R)})\}_p = \{(\nabla_{u*}\mathbf{F}^*)(A^* U_y^{*(T)})\}_p. \tag{58}$$

If we take account that the following relations are valid

$$\sum_{i=1}^{3} \Pi_i^{(R)} = \beta_1 r_0^{(1)} + \beta_2 r_0^{(2)} + \beta_3 r_0^{(3)}, \tag{59}$$

$$\sum_{i=1}^{3} \Pi_i^{*(T)} = \alpha_1 r_0^{*(1)} + \alpha_2 r_0^{*(2)} + \alpha_3 r_0^{*(3)}, \tag{60}$$

equation (58) implies eight algebraic equations for six coefficients.

If among these equations we can find precisely six linearly independent equations then we will be able to determine the unknown coefficients and the problem of the propagation of the weak discontinuity in $\mathscr{R}_u$ and $\mathscr{R}_{u*}$ will be solved.

In order to obtain the above mentioned algebraic system (58) we first deduce from (39) the following expression for ∇F.

$$\{\nabla_u \mathbf{F}\}_p = \left[\begin{array}{c|c} 0 & F_1 \\ \hline F_2 & 0 \end{array}\right] \tag{61}$$

with

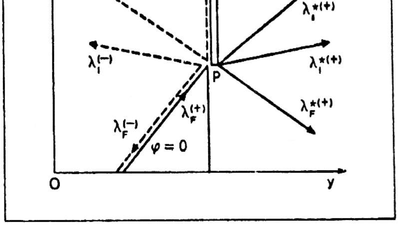

Fig. 1

$$F_1 = \begin{bmatrix} 0 & -B_1 & 0 \\ 0 & 0 & -B_1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad F_2 = \{A_2\} B_3 = 0 \\ w_3 = 0,$$

and a similar expression with starred variables for $\nabla_{u*}\mathbf{F}^*$.

From the jump conditions (56) across $\mathscr{D}$ we deduce that $B_1 = B_1^*$, so that the equations obtained from formula (58) are not all independent. The first is the same of the fourth, and the second is the same as the fifth, whilst the independent equations are:

$$\begin{cases} \lambda_F^2 \dfrac{K_{1F}\beta_1}{S} + \lambda_S^2 \dfrac{K_{1S}\beta_3}{S} - \lambda_F^{*2} \dfrac{K_{1F}^*\alpha_1}{S^*} - \\ \quad - \lambda_S^{*2} \dfrac{K_{1S}^*\alpha_3}{S^*} = - \lambda_F^2 \dfrac{K_{1S}}{S}, \\ \lambda_F^2\beta_1 + \lambda_S^2\beta_3 - \lambda_F^{*2}\alpha_1 - \lambda_S^{*2}\alpha_3 = -\lambda_F^2, \\ \lambda_F^3\beta_1 + \lambda_S^3\beta_3 + \lambda_F^{*3}\alpha_1 + \lambda_S^{*3}\alpha_3 = \lambda_F^3, \\ \lambda_F^3 \dfrac{K_{1F}\beta_1}{S} + \lambda_S^3 \dfrac{K_{1S}\beta_3}{S} + \\ \quad + \lambda_F^{*3} \dfrac{K_{1F}^*\alpha_1}{S^*} + \lambda_S^{*3} \dfrac{K_{1S}^*\alpha_3}{S^*} = \lambda_F^{*3} \dfrac{K_{1F}}{S}, \\ \lambda_I^2 BD\beta_2 - \lambda_I^{*2} B^*D^*\alpha_2 = 0, \\ \lambda_I^3 BD\beta_2 + \lambda_I^{*3} B^*D^*\alpha_2 = 0, \end{cases} \tag{62}$$

where, considering propagation in a constant state, we have chosen $\varphi_y = \varphi_{iy} = \varphi_{iy}^* = 1$.

From the last two we see immediately that if $\lambda_I + \lambda_I^* \neq 0$, then $\beta_2 = \alpha_2 = 0$. The solution for the first four equations is then

$$\beta_1 = \frac{\delta_1}{\delta}, \qquad \beta_3 = \frac{\delta_2\lambda_F^2}{\delta\lambda_S^2}, \\ \alpha_1 = \frac{\delta_3\lambda_F^2}{\delta\lambda_F^{*2}}, \qquad \alpha_3 = \frac{\delta_4\lambda_F^2}{\delta\lambda_S^{*2}} \tag{63}$$

with

$$\delta = \begin{vmatrix} \nu_F & \nu_S & -\nu_F^* & -\nu_S^* \\ 1 & 1 & -1 & -1 \\ \lambda_F & \lambda_S & \lambda_F^* & \lambda_S^* \\ \lambda_F\nu_F & \lambda_S\nu_S & \lambda_F^*\nu_F^* & \lambda_S^*\nu_S^* \end{vmatrix}, \\ \delta_1 = \begin{vmatrix} -\nu_F & \nu_S & -\nu_F & -\nu_S \\ -1 & 1 & -1 & -1 \\ \lambda_F & \lambda_S & \lambda_F^* & \lambda_S^* \\ \lambda_F\nu_F & \lambda_S\nu_S & \lambda_F^*\nu_F^* & \lambda_S^*\nu_S^* \end{vmatrix}, \\ \delta_2 = \begin{vmatrix} \nu_F & -\nu_F & -\nu_F^* & -\nu_S^* \\ 1 & -1 & -1 & -1 \\ \lambda_F & \lambda_F & \lambda_F^* & \lambda_S^* \\ \lambda_F\nu_F & \lambda_F\nu_F & \lambda_F^*\nu_F^* & \lambda_S^*\nu_S^* \end{vmatrix}, \tag{64}$$

$$\delta_3 = \begin{vmatrix} \nu_F & \nu_S & -\nu_F & -\nu_S^* \\ 1 & 1 & -1 & -1 \\ \lambda_F & \lambda_S & \lambda_F & \lambda_S^* \\ \lambda_F\nu_F & \lambda_S\nu_S & \lambda_F\nu_F & \lambda_S^*\nu_S^* \end{vmatrix},$$

$$\delta_4 = \begin{vmatrix} \nu_F & \nu_S & -\nu_F^* & -\nu_F \\ 1 & 1 & -1 & -1 \\ \lambda_F & \lambda_S & \lambda_F^* & \lambda_F \\ \lambda_F\nu_F & \lambda_S\nu_S & \lambda_F^*\nu_F^* & \lambda_F\nu_F \end{vmatrix}, \tag{64}$$

where $\nu = K/S$.

For the weak discontinuity transported along $\mathscr{D}$ we observe that if we denote by $\psi(x, t) = 0$ the equation of the line we have [11]:

$$\mathbf{U}^{*(T)} - \mathbf{U}^{(R)} = \boldsymbol{\chi} - \psi \boldsymbol{\Pi}_0, \tag{65}$$

where

$$\boldsymbol{\chi} = (\mathbf{U}^*)_{\psi^*} - (\mathbf{U})_{\psi}. \tag{66}$$

$$\boldsymbol{\Pi}_0 = \beta_4^{(1)} r_0^{(4,1)} + \beta_4^{(2)} r_0^{(4,2)} + \alpha_4^{(1)} r_0^{*(4,1)} + \alpha_4^{(2)} r_0^{*(4,2)}. \tag{67}$$

First we determine $\boldsymbol{\chi}$ by solving the equation [11]:

$$\frac{\mathrm{d}\boldsymbol{\chi}}{\mathrm{d}\tau} = \sum_{i=1}^{3} \lambda_i \boldsymbol{\Pi}_i^{(R)} + \boldsymbol{\Pi}\lambda_1 - \sum_{i=1}^{3} \lambda_i^* \boldsymbol{\Pi}_i^{*(T)}, \tag{68}$$

where τ is a time along $\mathscr{D}$.

After using this result in the equation

$$\sum_{i=1}^{3} (\lambda_F - \lambda_i^*) \boldsymbol{\Pi}_i^{*(T)} = \frac{\mathrm{d}\boldsymbol{\chi}}{\mathrm{d}t} + \sum_{i=1}^{3} (\lambda_F - \lambda_i) \boldsymbol{\Pi}_i^{(R)} + \lambda_1 \boldsymbol{\Pi}_0, \tag{69}$$

it follows by pre-multiplication, respectively, by $\mathbf{l}_0^{(4,1)}$, $\mathbf{l}_0^{(4,2)}$, $\mathbf{l}_0^{*(4,1)}$, $\mathbf{l}_0^{*(4,2)}$ that we obtain four algebraic equations for the unknown coefficients, though we do not state these explicitly here. This completes the solution of the problem of the transport of a weak discontinuity through a magneto-elastic medium.

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